Kuwait University — Math. 101 — July 29, 2000
Dept. of Math. & Comp. Sci. — Final Exam. — Duration: 2 hours

Calculators and mobile phones are not allowed.
Answer the following questions. Each question is worth 4 Points.

1. Find the following limit, if it exists
$$\lim_{x\to 2}\frac{\sin(x-2)}{x^2-4}.$$

2. Let
$$f(x)=\begin{cases}\dfrac{x^2+2x-3}{x^2-1}, & \text{if } x\neq\pm 1,\\[2ex] 2, & \text{if } x=1.\end{cases}$$
Find all discontinuities of f and classify them.

3. Show that the equation $x^7+5x+3=0$ has exactly one real root.

4. Let
$$f(x)=\begin{cases}A+2x, & \text{if } x\leqslant 1,\\[1ex] 2-Bx^2, & \text{if } x>1.\end{cases}$$
Find the constants A and B so that f is differentiable at $x=1$.

5. Find the equation of the normal line to the graph of the equation
$$\sqrt{x}+\sqrt{y}=3$$
at the point where $x=1$.

6. Find $f'(x)$, if
$$(a)\quad f(x)=\sin^5(x^3+1)^4,\qquad (b)\quad f(x)=\int_{x^3}^{\tan x}3t^3\,dt.$$

7. Evaluate the following integrals:
$$(a)\quad \int_0^{\pi/4}\tan^2 x\,dx,\qquad (b)\quad \int_{-1}^{1}\frac{x}{\sqrt{5-x^2}}dx.$$

8 Find the area of the region bounded by the graphs of the equations
$$x+y^2=4\quad\text{and}\quad x=0.$$

9. The region bounded by the graphs of the equations
$$y=x^2\quad\text{and}\quad x=y^2$$
is revolved about the line $y=-3$. Find the volume of the resulting solid.

10. Find the arc length of the graph of $y=x^{3/2}$ from $A(0,0)$ to $B(4,8)$.

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① $\lim_{x\to 2}\frac{\sin(x-2)}{x-2}=1$, $\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4}\Rightarrow \lim_{x\to 2}\frac{\sin(x-2)}{(x-2)(x+2)}=\frac{1}{4}$.

② At $x=1$: $\lim_{x\to 1}f(x)=\lim_{x\to 1}\frac{(x-1)(x+3)}{(x-1)(x+1)}=\lim_{x\to 1}\frac{x+3}{x+1}=2=f(1)$

$\therefore$ f is Continuous at $x=1$.

At $x=-1$: $\lim_{x\to -1^-}f(x)=\lim_{x\to -1^-}\frac{x+3}{x+1}=-\infty$, $\lim_{x\to -1^+}f(x)=\lim_{x\to -1^+}\frac{x+3}{x+1}=\infty$

f has infinite discontinuity at $x=-1$.

③ Let $f(x)=x^7+5x+3$. f is Cont. on $[-1,0]$, $f(-1)<0$ & $f(0)>0$

$\therefore$ f has a zero between -1 and 0 (by IVT) $\Rightarrow$ $f(x)=0$ has a Solution x_1. suppose that x_2 is another Solution; $x_2\neq x_1$, Consider the interval $[x_1,x_2]$ if $x_2>x_1$ (or $[x_2,x_1]$ if $x_1>x_2$)

f satisfies the hypotheses of Rolle's theorem on $[x_1,x_2]\Rightarrow$ there exists at least $c\in(x_1,x_2)$; $f'(c)=0$, but $f'(x)=7x^6+5$ is never zero $\Rightarrow x_2=x_1$ i.e. x_1 is unique.

($f'(x)>0$ for all $x\in\mathbb{R}\Rightarrow f$ is $\nearrow\Rightarrow$ the graph of f cuts the x-axis at most once $\Longrightarrow$ the solution is unique.)

④ f is differentiable at $x=1\Rightarrow f$ is Continuous at $x=1$

- Continuity: $\lim_{x\to 1^+}f(x)=f(1)\Rightarrow 2-B=A+2\Rightarrow \boxed{A=-B}$

- differentiability: $f'(1^+)=f'(1^-)$

$f'(1^+)=\lim_{x\to 1}\frac{2-Bx^2-A-2}{x-1}=-\lim_{x\to 1}\frac{Bx^2+A}{x-1}=-B\lim_{x\to 1}\frac{x^2-1}{x-1}$

$=-B\lim_{x\to 1}(x+1)=-2B$

$f'(1^-)=2$

$\Rightarrow -2B=2\Rightarrow B=-1$

$\Rightarrow A=1$

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(5) $x = 1 \Rightarrow y = 4$. Substitute in $x^{-\frac{1}{2}} + y^{-\frac{1}{2}} y' = 0 \Rightarrow 1 + \frac{1}{2} m = 0 \Rightarrow$

$m = -2$ the slope of TL at $P(1,4) \Rightarrow m_{\perp} = \frac{1}{2}$ the slope of the NL $\Rightarrow$ $y - 4 = \frac{1}{2}(x-1)$.

(6) a) $f'(x) = 5 \sin^4 (x^3+1)^4 . \cos (x^3+1)^4 . 4 (x^3+1)^3 . 3x^2$

$= 60 \; x^2 \; (x^3+1)^3 \; \sin^4 (x^3+1)^4 \; \cos (x^3+1)^4$.

b) $f'(x) = 3 \tan^3 x . \sec^2 x - 3x^9 (3x^2) = 3 \tan^3 x \sec^2 x - 9x^{11}$.

(7) a) $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \int \sec^2 x \, dx - \int dx = \tan x - x$

$\int_0^{\frac{\pi}{4}} \cdots = \left[\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right] - [\tan 0 - 0] = 1 - \frac{\pi}{4}$.

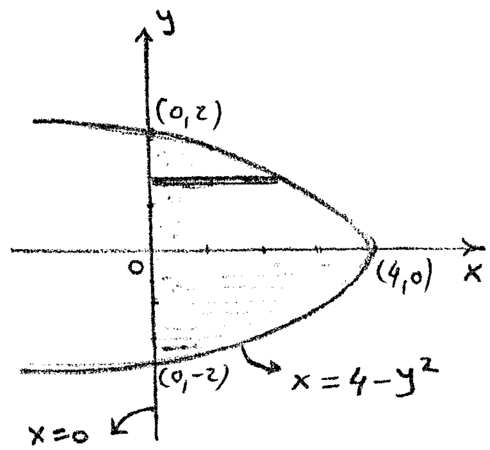
b) $\int_{-1}^{1} \frac{x}{\sqrt{5-x^2}} dx = 0$ since $\frac{x}{\sqrt{5-x^2}}$ is odd.

$\left(\int \frac{x}{\sqrt{5-x^2}} dx = -\sqrt{5-x^2} \Rightarrow \int_{-1}^{1} \cdots = -\sqrt{4} + \sqrt{4} = 0 \right)$

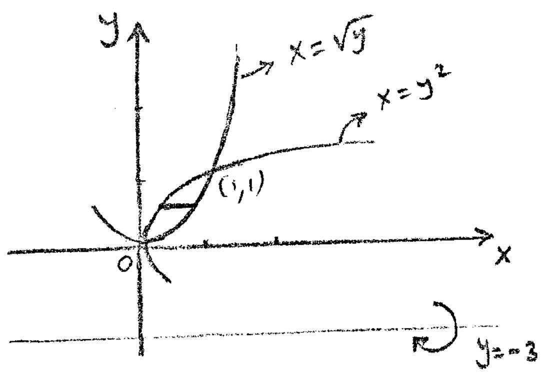
(8) $A = \int_{-2}^{2} (4-y^2) \, dy = 2 \int_0^2 (4-y^2) \, dy$

$= 2 \left[4y - \frac{1}{3} y^3 \right]_0^2$

$= 2 \left(8 - \frac{8}{3} \right) = \boxed{\frac{32}{3}}$



(9) <u>Cylindrical Shell Method:</u>



$al = \sqrt{y} - y^2$

$ar = y+3$

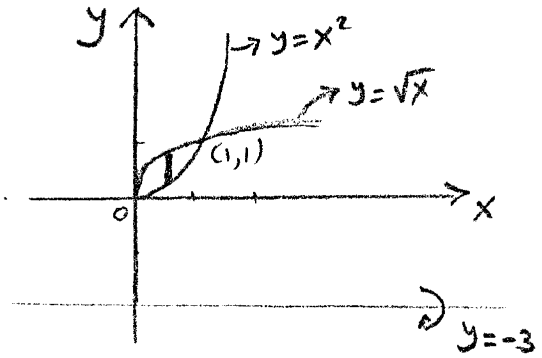
$th = dy \;;\quad y: 0 \to 1$

$$V = 2\pi \int_0^1 (\sqrt{y}-y^2)(y+3)\,dy = 2\pi \int_0^1 (y^{\frac{3}{2}} + 3y^{\frac{1}{2}} - y^3 - 3y^2)\,dy$$

$$= 2\pi \left[\frac{2}{5} y^{\frac{5}{2}} + 2y^{\frac{3}{2}} - \frac{1}{4} y^4 - y^3 \right]_0^1$$

$$= 2\pi \left(\frac{2}{5} + 2 - \frac{1}{4} - 1 \right) = \boxed{\frac{23}{10}\pi}$$

<u>Washer Method:</u>



$or = \sqrt{x} + 3$

$ir = x^2 + 3$

$th = dx \;;\quad x: 0 \to 1$

$$V = \pi \int_0^1 \left[(\sqrt{x}+3)^2 - (x^2+3)^2 \right] dx$$

$$= \pi \int_0^1 (x + 6x^{\frac{1}{2}} - x^4 - 6x^2)\,dx = \pi \left[\frac{1}{2}x^2 + 4x^{\frac{3}{2}} - \frac{1}{5}x^5 - 2x^3 \right]_0^1$$

$$= \pi \left(\frac{1}{2} + 4 - \frac{1}{5} - 2 \right) = \boxed{\frac{23\pi}{10}}$$

(10) $y = x^{\frac{3}{2}} \Rightarrow 1 + (y')^2 = 1 + \frac{9}{4}x$

$$L = \int_0^4 \sqrt{1 + \frac{9}{4}x}\;dx = \frac{4}{9}\,\frac{2}{3}\left(1 + \frac{9}{4}x\right)^{\frac{3}{2}} \Bigg|_{x=0}^{x=4}$$

$$= \frac{8}{27}\left[(10)^{\frac{3}{2}} - 1 \right] = \frac{8}{27}\left(10\sqrt{10} - 1 \right)$$